

# MINIMAL ZONOTOPES CONTAINING THE CROSSPOLYTOPE

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ABSTRACT. Motivated by the problem to improve Minkowski's lower bound on the successive minima for the class of zonotopes we determine the minimal volume of a zonotope containing the standard crosspolytope. It turns out that this volume can be expressed via the maximal determinant of a  $\pm 1$ -matrix, and that in each dimension the set of minimal zonotopes contains a parallelepiped. Based on that link to  $\pm 1$ -matrices, we characterize all zonotopes attaining the minimal volume in dimension 3 and present related results in higher dimensions.

## 1. INTRODUCTION

Let  $\mathbb{R}^n$  be the  $n$ -dimensional Euclidean space and let  $\mathbb{Z}^n$  be the integral lattice. The set of all 0-symmetric convex bodies, i.e., compact, convex sets, which are symmetric with respect to the origin is denoted by  $\mathcal{K}_0^n$ . Minkowski's second theorem on the successive minima of a body  $K \in \mathcal{K}_0^n$  states that [5, pp. 376]

$$(1.1) \quad \frac{2^n}{n!} \prod_{i=1}^n \frac{1}{\lambda_i(K)} \leq \text{vol}(K) \leq 2^n \prod_{i=1}^n \frac{1}{\lambda_i(K)},$$

where  $\text{vol}(K)$  denotes the volume of  $K$ , and  $\lambda_i(K)$  is the  $i$ -th successive minimum, i.e.,

$$\lambda_i(K) = \min\{t > 0 : \dim(tK \cap \mathbb{Z}^n) \geq i\}, \quad 1 \leq i \leq n.$$

In the context of relations between the successive minima and the roots of the Ehrhart polynomial of a lattice polytope [7], the problem arises to improve the lower bound for a special class of polytopes, namely zonotopes. A zonotope is the Minkowski sum of finitely many line segments and can be characterized by the property that all its two-dimensional faces are centrally-symmetric. For more properties on zonotopes we refer to the books [3, 13]. In contrast to the upper bound, the lower bound of (1.1) follows easily from considering a suitable crosspolytope inside the body  $K$ . Here we also want to improve this lower bound for the class of zonotopes and as a corollary of our main result we get:

**Corollary 1.1.** *Let  $Z \subset \mathbb{R}^n$  be a zonotope. Then*

$$\frac{2^n}{n^{n/2}} \prod_{i=1}^n \frac{1}{\lambda_i(Z)} \leq \text{vol}(Z).$$

Observe, that this bound is, roughly speaking,  $(\sqrt{n})^{n-1}$  larger than the one given in (1.1). For the proof we have to answer the question:

*What is the minimal volume of a zonotope containing  
the standard crosspolytope  $C_n^* = \text{conv}(\pm e_i : i = 1, \dots, n)$ ?*

Here  $e_i$  denotes the  $i$ -th coordinate unit vector. In order to present the answer we have to introduce maximal determinants of  $\pm 1$ -matrices. We define for every dimension  $n$

$$\text{maxdet}(n) = \max\{|\det(M)| : M \text{ } (n \times n)\text{-matrix with entries } \pm 1\}.$$

Obviously,  $\text{maxdet}(n) \leq n^{n/2}$  and the well known Hadamard conjecture states that

$$\text{maxdet}(4m) = (4m)^{2m}.$$

The conjecture is known to be true for all  $m \leq 166$ , but in general not much is known about  $\text{maxdet}(n)$  (cf. [10] for more information).

We will show the following relation between  $\text{maxdet}(n)$  and minimal volume  $C_n^*$  containing zonotopes.

**Theorem 1.2.** *Let  $Z \subset \mathbb{R}^n$  be a zonotope containing  $C_n^*$ . Then*

$$\text{vol}(Z) \geq \text{vol}(C_n^*) \frac{n!}{\text{maxdet}(n)}.$$

*Moreover, among all zonotopes of minimal volume containing  $C_n^*$  there exists always a parallelepiped.*

The proof of this theorem as well as of Corollary 1.1 will be given in the next section.

Unfortunately, we can not classify all zonotopes having minimal volume in all dimensions. In general, we will show that such a polytope has at most  $2^{n-1}$  generators (cf. Lemma 2.5), and in dimension 3 we will give a complete answer to that problem. To this end we denote for an  $(n \times m)$ -matrix  $A = (a_1, \dots, a_m)$ ,  $a_i \in \mathbb{R}^n$ , by  $Z(A) = \sum_{j=1}^m [-a_j, a_j]$  the 0-symmetric zonotope symmetrically generated by the  $m$  column vectors  $a_i$  of the matrix  $A$ . Here  $[x, y]$  denotes the convex hull of  $x, y \in \mathbb{R}^n$ .

In Section 3 we show the following characterization of minimal volume zonotopes containing  $C_3^*$ .

**Theorem 1.3.** *The set of 3-dimensional minimal volume zonotopes containing the crosspolytope is*

$$\left\{ Z \left( \left( \begin{array}{cccc} -t_1 & 1/2 - t_1 & 1/2 - t_1 & t_1 \\ 1/2 - t_2 & -t_2 & 1/2 - t_2 & t_2 \\ 1/2 - t_3 & 1/2 - t_3 & -t_3 & t_3 \end{array} \right) : (t_1, t_2, t_3)^\top \in T \right) \right\},$$

where  $T$  is the 3-dimensional simplex

$$T = \text{conv} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \right\}.$$

For a "visualisation" of this theorem, we refer to Section 3. Regarding the approximation of  $C_n^*$  by zonoids, i.e., bodies which can be arbitrarily well approximated by zonotopes, Rolf Schneider [14] posed the following related problem:

**Problem 1.4.** *For which zonoids  $Z$  is the factor  $\lambda$  with  $C_n^* \subseteq Z \subseteq \lambda C_n^*$  minimal?*

He proved a general lower bound on such a  $\lambda$  which in dimension 3 becomes  $\frac{3}{2}$ . By analyzing the zonotopes of Theorem 1.3 we found out that only the rhombic dodecahedron is a minimal volume zonotope containing  $C_3^*$  as well as a solution to Schneider's Problem in dimension 3 (cf. Proposition (3.2)). In dimensions  $\geq 4$  we conjecture that only parallelepipeds are minimal volume zonotopes containing  $C_n^*$ , i.e.,

**Conjecture 1.5.** *Let  $n \geq 4$  and let  $Z$  be a zonotope of minimal volume containing  $C_n^*$ . Then  $Z$  is a parallelepiped.*

In Lemma 4.4 we will give a sufficient criterion which implies the conjecture above. This criterion allows us to show that Conjecture 1.5 is true in dimensions  $n$  where the Hadamard conjecture is true (Corollary 4.5), and we also use it for verifying computationally the conjecture in dimensions 4 to 18, 20 and 21 (Proposition 4.6).

For similar questions regarding the approximation of arbitrary convex bodies by zonotopes or parallelepipeds we refer to [1], [6], [8], [9], [11].

## 2. MINIMAL ZONOTOPES AND SUCCESSIVE MINIMA

First we observe that such a minimal volume zonotope has to be indeed symmetric with respect to the origin:

**Proposition 2.1.** *Let  $K \subset \mathbb{R}^n$  be 0-symmetric. Every minimal volume zonotope containing  $K$  is 0-symmetric as well.*

*Proof.* Let  $Z = s + \sum_{j=1}^m [-a_j, a_j]$ ,  $m \geq n$ ,  $s, a_1, \dots, a_m \in \mathbb{R}^n$ , be a minimal volume zonotope containing  $K$  and let  $Z_0 := \sum_{j=1}^m [-a_j, a_j]$ . So we know  $K \subseteq s + Z_0$ , and we have to show that in the case  $s \neq 0$  there exists a zonotope containing  $K$  of smaller volume.

To this end we may assume  $s = e_n$  and let  $\gamma := \max\{x_n : x \in K\}$ , i.e.,  $\gamma$  is the maximal last coordinate of a point in  $K$ . By the 0-symmetry of  $K$  and  $Z_0$  we know that  $\pm e_n + K \subseteq Z_0$ . Hence for any point  $x \in K$  we have  $[-e_n + x, x + e_n] \subset Z_0$  and thus  $M(\gamma)x \in Z_0$ , where  $M(\gamma)$  is the  $(n \times n)$ -diagonal matrix with diagonal entries  $1, \dots, 1, 1 + \frac{1}{\gamma}$ . So we have

$$K \subseteq M(\gamma)^{-1} Z_0,$$

and since the right hand side is a zonotope of smaller volume than  $Z_0$  we have the desired contradiction.  $\square$

The next proposition mainly shows that all the vertices of  $C_n^*$  can be assumed to be vertices as well of a minimal volume zonotope. To this end we denote the vertices of a polytope  $P$  by  $\text{vert}(P)$ .

**Proposition 2.2.** *Let  $Z$  be a zonotope of minimal volume containing  $C_n^*$ . Then there is a linear transformation  $T$  with  $\det(T) = 1$  such that  $C_n^* \subseteq TZ$  and  $\text{vert}(C_n^*) \subseteq \text{vert}(TZ)$ .*

*Proof.* Let  $k = \frac{1}{2}\#(\text{vert}(C_n^*) \setminus \text{vert}(Z))$ . We prove this result by induction with respect to  $k$ . If  $k = 0$  we are done. Thus let  $k > 0$ . W.l.o.g. let  $\pm e_1 \notin \text{vert}(Z)$ . Since  $Z$  is of minimal volume,  $e_1 \in \text{bd}(Z)$  and let  $F$  be the smallest face of  $Z$  containing  $e_1$ . Then  $\gamma := \max\{x_1 : x \in F\} \geq 1$  and let  $v$  be a vertex with  $v_1 = \gamma$ . Define  $C = \text{conv}(\pm v, \pm e_2, \dots, \pm e_n)$ . Then  $\frac{1}{2}\#(\text{vert}(C) \setminus \text{vert}(Z)) < k$  and  $C \subseteq Z$ . Then for  $\bar{A} := (v, e_2, \dots, e_n)^{-1}$  we get  $\bar{A}C = C_n^*$  and  $\det(\bar{A}) = \frac{1}{\gamma}$  and thus  $\text{vol } \bar{A}Z \leq \text{vol } Z$ . Since  $Z$  is of minimal volume and  $C_n^* \subseteq \bar{A}Z$ , we get that  $\det(\bar{A}) = 1$  and thus  $\gamma = 1$ . This means that  $F$  is contained in  $\{x \in \mathbb{R}^n : x^\top e_1 = 1\}$  and thus that  $Z$  and  $\bar{A}Z$  have a generator which is orthogonal to  $e_1$ . Furthermore by construction  $C_n^*$  has at most  $2k - 2$  vertices that are not vertices of  $\bar{A}Z$ . Thus by induction hypothesis there exists a matrix  $A$  with  $\det(A) = 1$  such that  $\text{vert}(C_n^*) \subseteq \text{vert}(A\bar{A}Z)$ . Thus let  $T = A\bar{A}$ .  $\square$

**Remark 2.3.** *We want to point out that, if  $\text{vert}(C_n^*) \not\subseteq \text{vert}(Z)$ , the transformation  $T$  constructed in the proof above is a composition of transformations which fix  $n - 1$  of the coordinate unit vectors and at least one generator of  $Z$ . These generators are orthogonal to the remaining coordinate unit vector. This means that the zonotope  $TZ$  also has at least one generator, which is orthogonal to a coordinate unit vector.*

In the following we consider only such zonotopes, which have all vertices of  $C_n^*$  as vertices.

In order to simplify the notation we will denote for a matrix  $A \in \mathbb{R}^{n \times m}$  and a subset  $I \subset \{1, \dots, m\}$  by  $A_I$  the  $(n \times (\#I))$ -submatrix with columns indexed by the elements of  $I$  in increasing order. Analogously we denote for  $I \subset \{1, \dots, n\}$  by  $A^I$  the submatrix with rows indexed by elements of  $I$ .

Now we come to the proof of Theorem 1.2:

*Proof of Theorem 1.2.* We can assume that  $Z = \sum_{j=1}^m [-a_j, a_j]$  for  $m \geq n$  and  $a_1, \dots, a_m \in \mathbb{R}^n$  by Proposition 2.1. Let  $A$  be the  $(n \times m)$ -matrix  $(a_1, \dots, a_m)$ . By Proposition 2.2 we may assume that  $\text{vert}(C_n^*) \subseteq \text{vert}(Z) \subseteq \{\sum_{j=1}^m \pm a_j\}$ , i.e., all vertices of  $C_n^*$  can be written as  $\pm 1$ -combinations of  $a_1, \dots, a_m$ . Thus there is a  $\pm 1$ -matrix  $H$  such that  $I_n = A \cdot H$ . This yields

together with the Cauchy-Binet formula

$$\begin{aligned}
 (2.1) \quad 1 &= \det(I_n) = \det(A \cdot H) \\
 &= \sum_{I \subset \{1, \dots, m\}, \#I=n} \det(A_I) \det(H^I) \\
 &\leq \sum_{I \subset \{1, \dots, m\}, \#I=n} |\det(A_I)| |\det(H^I)| \\
 &\leq \sum_{I \subset \{1, \dots, m\}, \#I=n} |\det(A_I)| \maxdet(n) \\
 &= \maxdet(n) \frac{1}{2^n} \text{vol } Z = \frac{\maxdet(n)}{n!} \frac{\text{vol}(Z)}{\text{vol}(C_n^*)}.
 \end{aligned}$$

Obviously, for a  $\pm 1$ -matrix  $H$  with  $\det(H) = \maxdet(n)$  and  $A = H^{-1}$  we get equality.  $\square$

Since  $\maxdet(n) \leq n^{n/2}$  the theorem implies

$$\text{vol}(Z) \geq \text{vol}(C_n^*) \frac{n!}{n^{n/2}}$$

with equality if and only if there exists an  $(n \times n)$ -Hadamard matrix. We recall that a Hadamard matrix is an  $(n \times n)$ - $\pm 1$ -matrix whose columns are pairwise orthogonal. Hadamard matrices are known to exist in dimensions which are powers of 2 and as mentioned in the introduction it is conjectured that they exist in every dimension divisible by 4. The best of our knowledge, the best known general lower bounds on  $\maxdet(n)$  are of the type (cf. [2])

$$\maxdet(n) \geq n^{\frac{n}{2}(1-c/\ln n)},$$

where  $c$  is a certain positive constant. We want to remark that a similar bound follows immediately from Theorem 1.2. Since the unit ball  $B_n \supseteq C_n^*$  can be arbitrarily well approximated by zonotopes, we get by the theorem and Stirling's formula

$$\maxdet(n) \geq n! \frac{\text{vol}(C_n^*)}{\text{vol}(B_n)} = 2^n \frac{\Gamma(n/2 + 1)}{\pi^{n/2}} \sim \left( \sqrt{\frac{2}{\pi e}} \right)^n n^{n/2} = n^{\frac{n}{2}(1-\bar{c}/\ln n)},$$

for a certain positive constant  $\bar{c}$ . Here  $\Gamma(\cdot)$  denotes the  $\Gamma$ -function. Corollary 1.1 is an immediate consequence of Theorem 1.2:

*Proof of Corollary 1.1.* Let  $x_i \in \mathbb{Z}^n$ ,  $1 \leq i \leq n$ , be linearly independent vectors such that  $x_i \in \lambda_i(Z)Z$ . Then

$$C := \text{conv} \left( \pm \frac{x_1}{\lambda_1(Z)}, \dots, \pm \frac{x_n}{\lambda_n(Z)} \right) \subset Z,$$

and by Theorem 1.2 we get

$$\begin{aligned} \text{vol}(Z) &\geq \frac{n!}{\text{maxdet}(n)} \text{vol}(C) \\ &= \frac{n!}{\text{maxdet}(n)} \frac{2^n}{n!} \left| \det \left( \frac{x_1}{\lambda_1(Z)}, \dots, \frac{x_n}{\lambda_n(Z)} \right) \right| \\ &\geq \frac{2^n}{\text{maxdet}(n)} \prod_{i=1}^n \frac{1}{\lambda_i(Z)}. \end{aligned}$$

Since  $\text{maxdet}(n) \leq n^{n/2}$  (cf. [10]), the corollary follows.  $\square$

Next we want to study the equality case in Theorem 1.2. We recall that for  $A = (a_1, \dots, a_m) \in \mathbb{R}^{n \times m}$ ,  $Z(A)$  denotes the zonotope  $\sum_{j=1}^m [-a_j, a_j]$ , and we are interested in the problem for which  $n$  and  $m$  does there exist an  $(n \times m)$ -matrix  $A$  with  $\text{vol} Z(A) = \frac{2^n}{\text{maxdet}(n)}$  and  $C_n^* \subseteq Z(A)$ .

By Proposition 2.2 we may assume that  $\text{vert}(C_n^*) \subseteq \text{vert}(Z(A))$  and thus there are matrices  $A = (a_1, \dots, a_m)$  and  $H$  as in the proof of Theorem 1.2, i.e.,  $A \cdot H = I_n$ . Furthermore we can assume that no two vectors of  $\{a_1, \dots, a_m\}$  are linearly dependent, because otherwise we can sum them up to one generator of the zonotope.

**Proposition 2.4.** *Let  $A$  be an  $(n \times m)$ -matrix such that  $Z(A)$  is a minimal volume zonotope containing  $C_n^*$  and let  $H$  be an  $(m \times n)$ - $\pm 1$ -matrix such that  $A \cdot H = I_n$ .*

- (i) *Let  $I \subseteq \{1, \dots, m\}$  with  $\#I = n$ . If  $\det(A_I) \neq 0$  then  $|\det(H^I)| = \text{maxdet}(n)$  and  $\text{sign}(\det(H^I)) = \text{sign}(\det(A_I))$ .*
- (ii) *If for some  $J \subseteq \{1, \dots, m\}$  the rows of  $H^J$  are linearly dependent, then the columns of  $A_J$  are linearly dependent as well.*

*Proof.* The equality in the first inequality in (2.1) in the proof of Theorem 1.2 is attained only if  $\text{sign}(\det(H^I)) = \text{sign}(\det(A_I))$  and the equality in the second inequality in (2.1) is attained only if  $|\det(H^I)| = \text{maxdet}(n)$  for all  $I$  such that  $\det(A_I) \neq 0$ . Thus we get part (i). For part (ii) we may assume  $J = \{1, \dots, k\}$  and  $k < n$ . From part (i) we conclude that  $\det(A_I) = 0$  for all  $I$  such that  $J \subseteq I$ . Now assume that the columns of  $A_J$  are linearly independent. Since the rank of  $A$  is  $n$  we can find an index set  $I^* \supseteq J$  such that the columns of  $A_{I^*}$  are linearly independent, which is a contradiction.  $\square$

By Proposition 2.4 we may assume that a matrix  $H$  of a minimal volume zonotope cannot have two linearly dependent rows, because otherwise the two generators of  $Z$  are linearly dependent. Hence  $H$  cannot have more than  $2^{n-1}$  rows and thus

**Lemma 2.5.** *Let  $Z$  be a minimal volume zonotope containing  $C_n^*$ . Then  $Z$  has at most  $2^{n-1}$  pairwise linearly independent generators.*

## 3. DIMENSION 3

The pictures in this section are made by `polymake` [4] and `JavaView` [12]. The aim of this section is to prove and explain Theorem 1.3. We call two  $\pm 1$ -matrices equivalent if one can be obtained from the other by a series of permutations and negations of rows and columns. Let  $A$  and  $H$  be as above. Negating a column of  $H$  coincides with negating a row of  $A$ , which corresponds to a reflexion of  $Z(A)$  with respect to a coordinate hyperplane. Negating a row of  $H$  coincides with negating a column of  $A$  which does not change  $Z(A)$  at all. Interchanging two columns of  $H$  coincides with interchanging two rows of  $A$ , which corresponds to a reflexion of  $Z(A)$  with respect to a hyperplane of the form  $\{x \in \mathbb{R}^n : x_i - x_j = 0\}$ , which does not change the 3-dimensional crosspolytope. Interchanging two rows of  $H$  does not change  $Z(A)$  at all.

*Proof of Theorem 1.3.* First we just consider inequivalent matrices  $H$ . By Lemma 2.5 we can only have 3 or 4 generators. First we consider the case  $m = 3$ . In this case  $H$  is a quadratic  $\pm 1$ -matrix with maximal determinant. Up to equivalence, the maximal determinant  $(3 \times 3)$ - $\pm 1$ -matrix

$H = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$  is unique (cf. [10]). The corresponding matrix of

generators  $A$  is  $A = H^{-1} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$  and the maximal determi-

nant is  $\max \det(3) = 4$ .

Now we consider the case  $m = 4$  and we assume that  $A$  does not have two linearly dependent columns. By Proposition 2.4,  $H$  cannot have two linearly dependent rows. Thus we can choose 4 of the 8 possible  $\pm 1$ -vectors in such a way that no two opposite vectors are chosen. Hence also in this case  $H$  needs to be unique up to equivalence, namely

$$H = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then we can solve the linear system  $A \cdot H = I_3$  and get solutions

$$A(t) = \begin{pmatrix} -t_1 & 1/2 - t_1 & 1/2 - t_1 & t_1 \\ 1/2 - t_2 & -t_2 & 1/2 - t_2 & t_2 \\ 1/2 - t_3 & 1/2 - t_3 & -t_3 & t_3 \end{pmatrix}$$

for all vectors  $t = (t_1, t_2, t_3)$  of 3 parameters. One can easily check that all  $(3 \times 3)$ -subdeterminants of  $H$  are  $\pm 4$ . For equality in (2.1) we need that  $\det(A(t)_I) = 0$  or  $\text{sign}(\det(A(t)_I)) = \text{sign}(\det(H^I))$  for all  $I$ :

I	$\det(H^I)$	$\det(A(t)_I)$
$\{1, 2, 3\}$	4	$\frac{1}{4}(-t_1 - t_2 - t_3 + 1)$
$\{1, 2, 4\}$	4	$\frac{1}{4}(t_1 + t_2 - t_3)$
$\{1, 3, 4\}$	-4	$\frac{1}{4}(-t_1 + t_2 - t_3)$
$\{2, 3, 4\}$	4	$\frac{1}{4}(-t_1 + t_2 + t_3)$

These conditions yield the following inequalities on  $t_i$ :

$$\begin{aligned} +t_1 + t_2 + t_3 &\leq 1 \\ +t_1 + t_2 - t_3 &\geq 0 \\ +t_1 - t_2 + t_3 &\geq 0 \\ -t_1 + t_2 + t_3 &\geq 0 \end{aligned}$$

Observe that  $t_1 = t_2 = t_3 = 0$  is allowed and coincides with the case  $m = 3$ . It is easy to see that the simplex  $T$  defined by the inequalities above has the vertices given in Theorem 1.3 (see Figure 1).

Let  $v_0, \dots, v_3$  be the vertices of  $T$  and let

$$A(t) := \begin{pmatrix} -t_1 & 1/2 - t_1 & 1/2 - t_1 & t_1 \\ 1/2 - t_2 & -t_2 & 1/2 - t_2 & t_2 \\ 1/2 - t_3 & 1/2 - t_3 & -t_3 & t_3 \end{pmatrix}$$

for all  $t = (t_1, t_2, t_3) \in T$ .

Thus it remains to show that the set given in Theorem 1.3 also contains the zonotopes corresponding to equivalent transformations of  $H$ , i.e., we have to show that the set is closed under negations of rows (i.e., reflections of the zonotope with respect to coordinate hyperplanes) and interchanges of rows of the generator matrix. The latter is easily seen, since  $t_1, t_2$  and  $t_3$  can be interchanged arbitrarily. Furthermore the transformations  $(t_1, t_2, t_3) \mapsto (t_1, 1/2 - t_2, 1/2 - t_3)$ ,  $(t_1, t_2, t_3) \mapsto (1/2 - t_1, t_2, 1/2 - t_3)$  and  $(t_1, t_2, t_3) \mapsto (1/2 - t_1, 1/2 - t_2, t_3)$  reflect the zonotope with respect to the hyperplanes  $x_1 = 0$ ,  $x_2 = 0$  and  $x_3 = 0$ , respectively.  $\square$

**Remark 3.1.** *Together with Remark 2.3 one can show that indeed, this set contains also all zonotopes of minimal volume which do not contain all vertices of  $C_3^*$  as vertices. To do this, we have to find rows in  $A(t)$  which are orthogonal to one of the coordinate unit vectors, say  $e_j$ . This only happens if  $t$  is contained in an edge of  $T$  and in this case,  $e_j$  is contained in an edge of  $Z(A(t))$ . The given set contains all possible "movements" of  $e_j$  along this edge (cf. Figure 2).*

In any facet-defining inequality of  $T$  equality is attained iff  $\det(A(t)_I) = 0$  only for the corresponding  $I$ . Thus  $t$  being a vertex of  $T$  means that all but one of the  $(3 \times 3)$ -subdeterminants of  $A(t)$  are zero, i.e., one of the columns of  $A(t)$  equals zero. Thus  $Z(A(t))$  is a parallelepiped (cf. Figure 2,  $t = t^0 = v_0$ ). For  $t$  lying in the interior of an edge of  $T$  exactly two of the subdeterminants are 0 and thus, there are two linearly dependent generators, i.e., the zonotope is a parallelepiped as well (cf. Figure 2,  $t = t^1 = \frac{1}{2}(v_0 + v_1)$ ). In the interior



FIGURE 1. The simplex  $T$  in Theorem 1.3

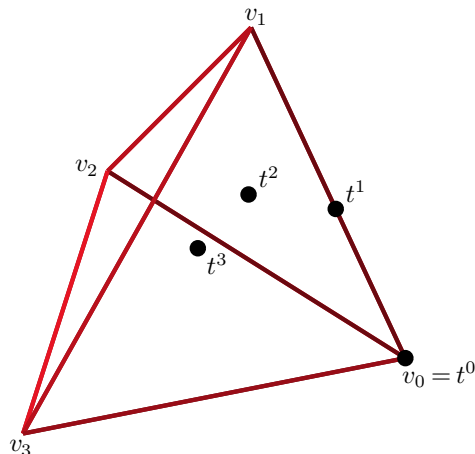
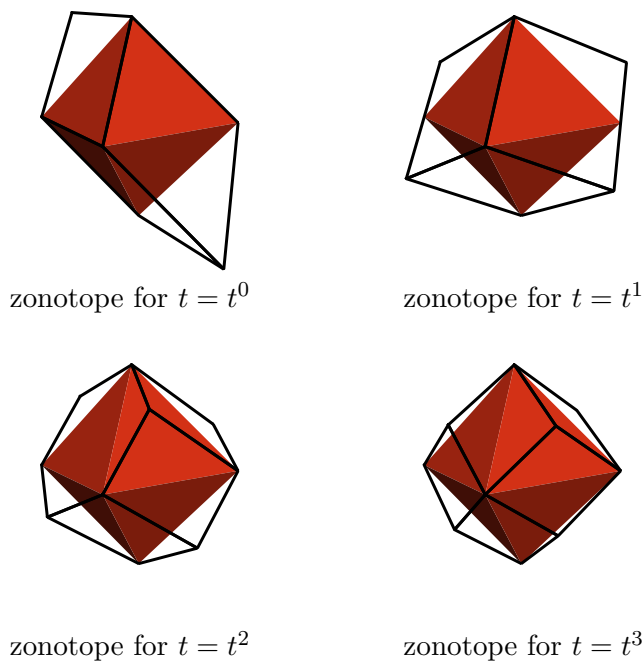


FIGURE 2. Minimal zonotopes in dimension 3 containing  $C_3^*$



of a facet exactly one equality is attained, i.e., the corresponding zonotope is a prism over a 6-gon (cf. Figure 2,  $t = t^2 = \frac{1}{3}(v_0 + v_1 + v_2)$ ). The interior points of  $T$  correspond to a zonotope with 4 generators in general position (cf. Figure 2,  $t = t^3 = \frac{1}{4}(v_0 + v_1 + v_2 + v_3)$ ), the rhombic dodecahedron. Observe that all these zonotopes are space-filling polytopes.

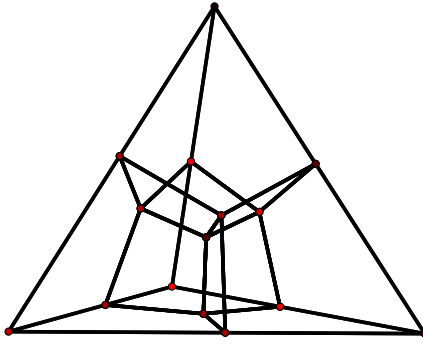
Next we want to study our minimal zonotopes in view of Schneider's question stated in Problem 1.4. Schneider showed in [14] that any  $\lambda$  with  $C_n^* \subseteq Z \subseteq \lambda C_n^*$ , where  $Z$  is a zonoid, satisfies

$$\lambda \geq 2^{-n+1} n \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} \sim \sqrt{\frac{2}{\pi}} \sqrt{n}.$$

For  $n = 3$  this reduces to  $\lambda \geq \frac{3}{2}$ , and next we calculate the factor  $\lambda$  for an arbitrary  $Z$  in the set given in Theorem 1.3.

To this end we consider the following subdivision of the simplex  $T$  in Figure 3. Here the dividing planes are defined by midpoints of edges of  $T$ , facets

FIGURE 3. A division of the simplex  $T$



of  $T$  and  $T$  itself. In formulas, we consider the following 4 parts:

- $T_1 := \{(t_1, t_2, t_3) \in T : t_1 \leq t_2, t_1 \leq t_3, t_2 + t_3 \geq \frac{1}{2}\}$
- $T_2 := \{(t_1, t_2, t_3) \in T : t_2 \leq t_3, t_2 \leq t_1, t_1 + t_3 \geq \frac{1}{2}\}$
- $T_3 := \{(t_1, t_2, t_3) \in T : t_3 \leq t_2, t_3 \leq t_1, t_1 + t_2 \geq \frac{1}{2}\}$
- $T_4 := \{(t_1, t_2, t_3) \in T : t_1 + t_2 \leq \frac{1}{2}, t_1 + t_3 \leq \frac{1}{2}, t_2 + t_3 \leq \frac{1}{2}\}$

In each part, the minimal value of the factor  $\lambda$  for  $Z(A(t_1, t_2, t_3))$  is affinely linear in  $(t_1, t_2, t_3)$ , i.e., for  $(t_1, t_2, t_3) \in T_i$  we get the following minimal values for  $\lambda$ :

$i$	$\lambda$
1	$1 + 2(-t_1 + t_2 + t_3)$
2	$1 + 2(+t_1 - t_2 + t_3)$
3	$1 + 2(+t_1 + t_2 - t_3)$
4	$3 - 2(+t_1 + t_2 + t_3)$ .

In particular we get the following values for the vertices of the division:

vertices of $T$	parallelepiped	$\lambda = 3$
midpoints of edges of $T$	parallelepiped	$\lambda = 2$
midpoints of facets of $T$	prism over a 6-gon	$\lambda = 5/3$
midpoint of $T$	rhombic dodecahedron	$\lambda = 3/2$ .

**Proposition 3.2.** *Only the rhombic dodecahedron is a minimal volume zonotope containing  $C_3^*$  as well as a solution to Problem 1.4 in dimension 3.*

#### 4. ARBITRARY DIMENSIONS

Next we look at arbitrary dimension  $n$ . First we consider the special case of zonotopes with generators in general position. The generators of an  $n$ -dimensional zonotope are said to be in general position, if any  $n$  of them are linearly independent.

**Lemma 4.1.** *In even dimensions among all zonotopes that contain the crosspolytope and whose generators are in general position only parallelepipeds have minimal volume.*

*Proof.* Let  $m > n$  and let  $A$  be an  $(n \times m)$ -matrix such that  $Z(A)$  is a minimal volume zonotope containing the  $n$ -dimensional crosspolytope and let the generators of  $A$  be in general position. Furthermore let  $H$  be the corresponding  $(m \times n)$ - $\pm 1$ -matrix with  $A \cdot H = I_n$  and let the rows of  $H$  be denoted by  $h_i$ . By Proposition 2.4 we get that  $\det(H^I) = \max \det(n)$  for all  $I \subset \{1, \dots, m\}$  with  $\#I = n$ . Thus  $h_{n+1}$  is a  $\pm 1$ -combination of  $h_1, \dots, h_n$  since otherwise replacing  $h_i, i \in \{1, \dots, n\}$ , by  $h_{n+1}$  in  $H_{\{1, \dots, n\}}$  the absolute value of the determinant would change. But this is a contradiction, since a  $\pm 1$ -combination of  $h_1, \dots, h_n$  is a vector with even entries in even dimensions. □

To deal with the general case, we consider the following lemma.

**Lemma 4.2.** *Let  $A = (a_1, \dots, a_{n+1})$  be an  $(n \times (n + 1))$ -matrix of rank  $n$ . Furthermore let  $a_j \neq 0$  for all  $j$  and let  $\det(A_{\{1, \dots, n\}}) \neq 0$ . Assume that at most  $k$  of the  $(n \times n)$ -subdeterminants of  $A$  are not 0. Then  $a_{n+1}$  is contained in the linear hull of  $k - 1$  columns of  $A_{\{1, \dots, n\}}$ .*

*Proof.* Since  $a_1, \dots, a_n$  are linearly independent, we have for  $I, J \subset \{1, \dots, n\}$ :

$$(4.1) \quad \text{lin}\{a_j : j \in I\} \cap \text{lin}\{a_j : j \in J\} = \text{lin}\{a_j : j \in I \cap J\}.$$

To see this let  $x \in \text{lin}\{a_j : j \in I\} \cap \text{lin}\{a_j : j \in J\}$ , i.e.,

$$x = \sum_{j \in I} \alpha_j a_j = \sum_{j \in J} \beta_j a_j.$$

Thus

$$0 = \sum_{j \in I \setminus J} \alpha_j a_j + \sum_{j \in I \cap J} (\alpha_j - \beta_j) a_j - \sum_{j \in J \setminus I} \beta_j a_j.$$

Hence  $\alpha_j = 0$  for  $j \in I \setminus J$  and  $\beta_j = 0$  for  $j \in J \setminus I$  and thus  $x \in \text{lin}\{a_j : j \in I \cap J\}$ .

By assumption there exists an index set  $S \subset \{1, \dots, n\}$ ,  $\#S = n + 1 - k$ , with  $\det(A_{\{1, \dots, n+1\} \setminus \{s\}}) = 0$  for all  $s \in S$ . Again since  $a_1, \dots, a_n$  are linearly independent we get that  $a_{n+1} \in \text{lin}\{a_j : j \in \{1, \dots, n\} \setminus \{s\}\}$  for all  $s \in S$ . Together with (4.1) above  $a_{n+1} \in \text{lin}\{a_j : j \in \{1, \dots, n\} \setminus S\}$  which completes the proof.  $\square$

**Remark 4.3.** *In particular we see that for such a matrix as in the lemma above it holds: If  $\det(A_I) = 0$  for all  $I \neq \{1, \dots, n+1\} \setminus \{l\}, \{1, \dots, n+1\} \setminus \{j\}$  then  $a_l$  is a multiple of  $a_j$ .*

**Lemma 4.4.** *If every  $((n+1) \times n)$ - $\pm 1$ -matrix has at most two  $(n \times n)$ -subdeterminants that are  $\text{maxdet}(n)$ , then all minimal volume zonotopes containing the  $n$ -dimensional crosspolytope are parallelepipeds.*

*Proof.* Assume all  $((n+1) \times n)$ - $\pm 1$ -matrices have at most two subdeterminants that are  $\text{maxdet}(n)$ . Let  $A = (a_1, \dots, a_m)$  be an  $(n \times m)$ -matrix with  $m > n$  such that  $Z(A)$  is a minimal volume zonotope containing the  $n$ -dimensional crosspolytope. Furthermore let  $H$  be the corresponding  $(m \times n)$ - $\pm 1$ -matrix with  $A \cdot H = I_n$ . W.l.o.g. we assume that  $\det(a_1, \dots, a_n) \neq 0$ . Now we consider  $A_{\{1, \dots, n, k\}}$ ,  $k > n$ . Since at most two subdeterminants of  $H_{\{1, \dots, n, k\}}$  are  $\text{maxdet}(n)$ , at most two of the subdeterminants of  $A$  are not 0. Thus, by Remark 4.3, there exists a  $j(k) \in \{1, \dots, n\}$  such that  $a_k$  is a multiple of  $a_{j(k)}$ . Since this is true for every  $k > n$ ,  $A$  consists of  $n$  linearly independent generators and some multiples of them. This means that  $Z(A)$  is a parallelepiped.  $\square$

**Corollary 4.5.** *In dimensions where Hadamard matrices exist, all minimal volume zonotopes containing the  $n$ -dimensional crosspolytope are parallelepipeds.*

*Proof.* Let  $n$  be a dimension where Hadamard matrices exist and let

$$H = \begin{pmatrix} h_1 \\ \dots \\ h_{n+1} \end{pmatrix}$$

be an  $((n+1) \times n)$ - $\pm 1$ -matrix. W.l.o.g. let  $\det(H_{\{1, \dots, n\}}) = \det(H_{\{2, \dots, n+1\}}) = \text{maxdet}(n)$ . Thus  $H_{\{1, \dots, n\}}$  and  $H_{\{2, \dots, n+1\}}$  are Hadamard matrices and hence,  $h_1$  and  $h_{n+1}$  are orthogonal to  $\text{lin}\{h_2, \dots, h_n\}$ . It follows that  $h_1$  and  $h_{n+1}$  are linearly dependent, and thus all other subdeterminants are zero. The claim follows by Lemma 4.4.  $\square$

**Proposition 4.6.** *For dimensions 4 to 18, 20 and 21 all  $((n+1) \times n)$ - $\pm 1$ -matrices have at most two  $(n \times n)$ -subdeterminants that are  $\text{maxdet}(n)$ . Thus in these dimensions all minimal volume zonotopes containing the  $n$ -dimensional crosspolytope are parallelepipeds.*

*Proof.* The computations in this proof were performed using Maple<sup>TM</sup>. To check, whether there exists an  $((n+1) \times n)$ - $\pm 1$ -matrix that has more than two  $(n \times n)$ -subdeterminants that are  $\maxdet(n)$  we used the following computational approach: To every  $(n \times n)$ - $\pm 1$ -matrix with maximal determinant, we append every  $\pm 1$ -vector as  $(n+1)$ st row. Thereby we construct every  $((n+1) \times n)$ - $\pm 1$ -matrix with at least one subdeterminant being  $\maxdet(n)$ . For all these matrices we calculate, how many subdeterminants are  $\maxdet(n)$ . The following Maple<sup>TM</sup>-code does this for a previously given maximal  $(n \times n)$ - $\pm 1$ -matrix  $H$ :

```
with(LinearAlgebra):
maxdet:=abs(Determinant(H));
one:=Vector(n,1):
for i from 0 to 2^(n-1)-1 do
  maxdet_set:={}:
  zero_one_vector:=convert(Bits[Split](i,bits=n),Vector);
  H0:=ScalarMultiply(zero_one_vector,2)-one;
  B:=<H0|H>;
  for k from 0 to n do
    if abs(Determinant(DeleteColumn(B,k+1)))=maxdet then
      maxdet_set:=maxdet_set union {k}:
    end if;
  end do;
if nops(maxdet_set)>2 then print(i,maxdet_set); end if;
end do;
```

Thus getting no output means that the matrix  $H$  cannot be submatrix of an  $((n+1) \times n)$ - $\pm 1$ -matrix that has more than two  $(n \times n)$ -subdeterminants that are  $\maxdet(n)$ . Hence we need to do this for all different (inequivalent)  $(n \times n)$ - $\pm 1$ -matrices with maximal determinant. For example for dimension 5 there is only one of these matrices and the input file for this matrix  $H$  is the following:

```
n:=5:
H:=<<-1,+1,+1,+1,+1>|
<+1,-1,+1,+1,+1>|
<+1,+1,-1,+1,+1>|
<+1,+1,+1,-1,+1>|
<+1,+1,+1,+1,-1>>:
```

We did this for all maximal matrices given in [10] from dimension 4 to 21. Unfortunately in dimension 19 the maximal determinant is not known.  $\square$

## REFERENCES

- [1] E.M. Bronstein, *Approximation of convex sets by polytopes*, Journal of Mathematical Sciences **153** (2008), 727–762.

- [2] G. F. Clements and B. Lindström, *A Sequence of  $(\pm 1)$ -Determinants with Large Values*, Proceedings of the American Mathematical Society, **16**(3) (1965), 548–550.
- [3] R. Gardner, *Geometric tomography*, Cambridge University Press, 1995.
- [4] E. Gawrilow and M. Joswig, *Geometric reasoning with polymake*, arXiv:math.CO/0507273, 2005, [www.opt.tu-darmstadt.de/polymake/](http://www.opt.tu-darmstadt.de/polymake/).
- [5] P. M. Gruber, *Convex and Discrete Geometry*, Springer, 2007.
- [6] L. J. Guibas, A. Nguyen, and L. Zhang, *Zonotopes as Bounding Volumes*, Proceedings of the fourteenth annual ACM-SIAM symposium on Discrete Algorithms (2003), 803–812.
- [7] M. Henk, A. Schürmann, and J. M. Wills, *Ehrhart polynomials and successive minima*, Mathematika **52** (2005), 1–16.
- [8] M. Hudelson, V. Klee, and D. Larman, *Largest  $j$ -Simplices in  $d$ -Cubes: Some Relatives of the Hadamard Maximum Determinant Problem*, Linear Algebra and its Applications **241-243** (1996), 519–598.
- [9] R.N. Karasev, *Inscribing a regular crosspolytope*, arXiv:0905.2671v1 [math.AT], May 2009, <http://arxiv.org/abs/0905.2671v1>.
- [10] W. Orrick and B. Solomon, *The Hadamard maximal determinant problem*, <http://www.indiana.edu/~maxdet>.
- [11] A. Pelczynski and S.J. Szarek, *On parallelepipeds of minimal volume countaining a convex symmetric body in  $\mathbb{R}^n$* , Math. Proc. Camb. Phil. Soc. **109** (1991), 125–148.
- [12] K. Polthier, K. Hildebrandt, E. Preuß, and U. Reitebuch, *JavaView - Interactive 3D Geometry and Visualization*, <http://www.javaview.de/>.
- [13] R. Schneider, *Convex bodies: The Brunn-Minkowski Theory*, Cambridge University Press, 1993.
- [14] ———, *On the Busemann Area in Minkowski Spaces*, Beitrge zur Algebra und Geometrie **42**, No. 1 (2001), 263–273.

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